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APROKSIMACIJA FUNKCIJA

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$$f(x) \approx Q(x) = \sum_{i=0}^{n} c_i g_i(x)$$

Definisati *oblik* i *bliskost* aproksimacije

1. INTERPOLACIJA

$$Q(x_k) = f(x_k), \quad k = 0, \ldots, n.$$

Lagrange-ov interpolacioni polinom

$$g_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

$$g_i(x_k) = \delta(i-k), \quad i, k = 0, \ldots, n, \longrightarrow c_i = f(x_i)$$

$$L_n(x) = \sum_{i=0}^{n} f(x_i) g_i(x) = \sum_{i=0}^{n} \left(\prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \right) f(x_i)$$

Greška interpolacije:

$$f(\overline{x}) - L_n(\overline{x}) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(\overline{x})$$

$$\omega_{n+1}(x) = \prod_{j=0}^{n}(x - x_j)$$

$$|f(\overline{x}) - L_n(\overline{x})| \leq \frac{1}{(n+1)!} \max |f^{(n+1)}| \, |\omega_{n+1}(\overline{x})|$$

Primer:

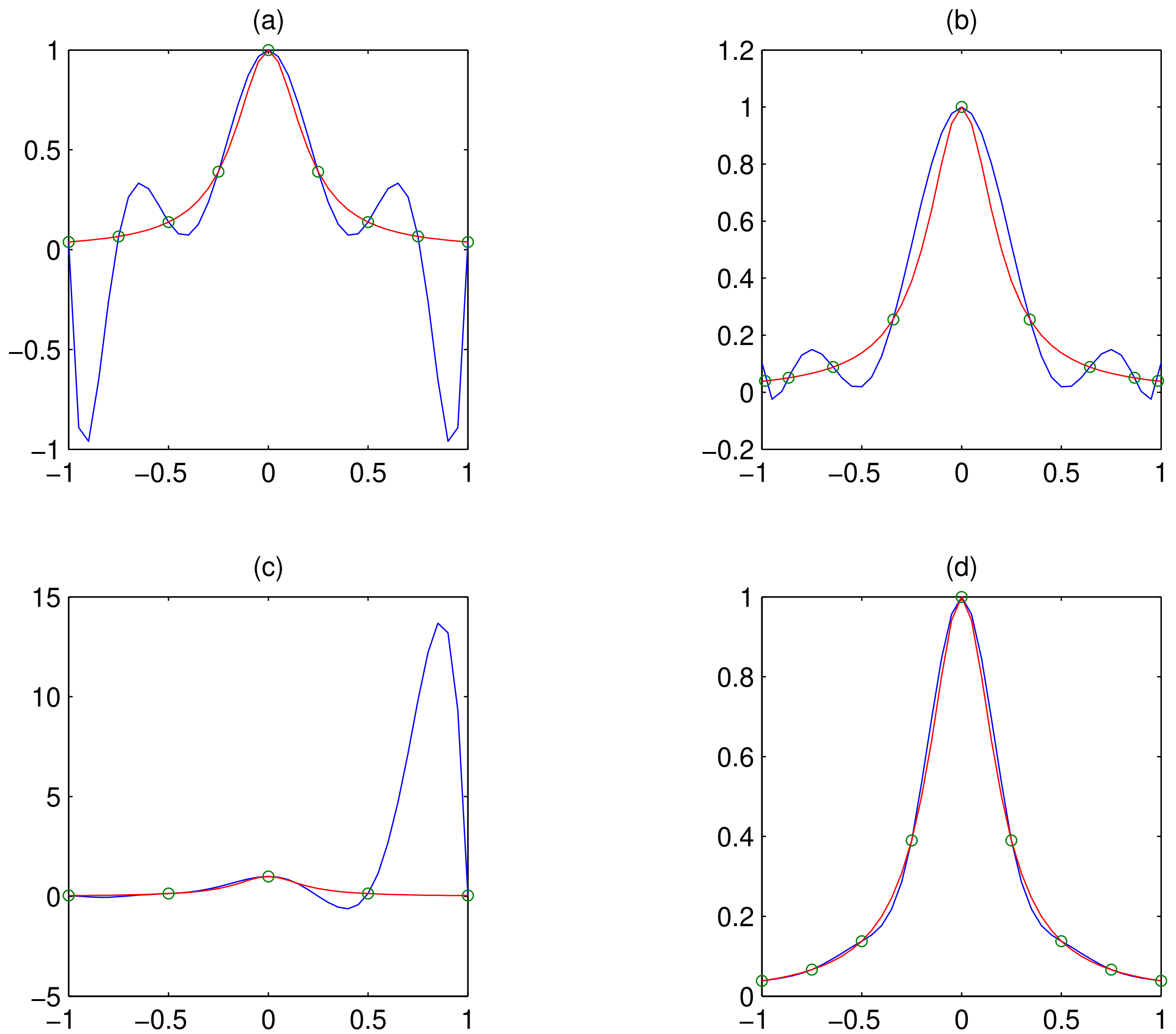
$$f(x) = \frac{1}{1+25x^2}, \quad x \in [-1,1], \quad n = 8$$

(a) Lagrange-ov polinom sa ekvidistantnim čvorovima

(b) Lagrange-ov polinom sa čvorovima koji su nule Čebišev-ljevog polinoma

(c) Hermite-ov polinom $(2,3,2,1,1)$

(d) Splajn



Interpolacija

Hermite-ov interpolacioni polinom

$$Q(x) \equiv H_n(x), \qquad n = \sum_{i=0}^{m} n_i - 1,$$

$$H_n^{(k)}(x_i) = f^{(k)}(x_i), \qquad \begin{array}{c} k = 0, \ldots, n_i - 1, \\ i = 0, \ldots, m. \end{array}$$

Splajn

$$\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$$

Splajn $S_\Delta^m(x)$ reda m definisan podelom Δ je realna funkcija na intervalu $[a, b]$, koja ima sledeće osobine

(*i*) $S_\Delta^m \in \mathcal{C}^{m-1}[a, b]$,

(*ii*) S_Δ^m je polinom stepena m na svakom od intervala $[x_i, x_{i+1}]$, $i = 0, \ldots, n-1$, podele Δ.

Interpolacioni kubni splajn, $x \in [x_i, x_{i+1}]$

$$S_\Delta(f;x) = \alpha_i + \beta_i(x - x_i) + \gamma_i(x - x_i)^2 + \delta_i(x - x_i)^3,$$

$$\begin{aligned}
\alpha_i &= S_\Delta(f;x_i) = f(x_i) \equiv f_i \\
\beta_i &= S'_\Delta(f;x_i) = \frac{f_{i+1} - f_i}{h_{i+1}} - \frac{h_{i+1}}{6}(2M_i + M_{i+1}) \\
\gamma_i &= \tfrac{1}{2} S''_\Delta(f;x_i) = \tfrac{1}{2} M_i \\
\delta_i &= \tfrac{1}{6} S'''_\Delta(f;x_i) = \frac{1}{6h_{i+1}}(M_{i+1} - M_i).
\end{aligned}$$

$h_i = x_i - x_{i-1}$, moment splajna $M_i = S''_\Delta(f;x_i)$

$$\mu_i M_{i-1} + 2M_i + \nu_i M_{i+1} = \lambda_i, \quad i = 1, \ldots, n-1,$$

$$\nu_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \quad \mu_i = \frac{h_i}{h_i + h_{i+1}} = 1 - \nu_i,$$

$$\begin{aligned}
\lambda_i &= \frac{6}{h_i + h_{i+1}}\left(\frac{f_{i+1} - f_i}{h_{i+1}} - \frac{f_i - f_{i-1}}{h_i}\right) \\
&= 6f[x_{i-1}, x_i, x_{i+1}]
\end{aligned}$$

Određen je uslovima interpolacije

$$S_\Delta(f;x_i) = f(x_i), \quad i = 0,\ldots,n,$$

i uslovima na krajevima intervala
(*i*) $S''_\Delta(f;a) = S''_\Delta(f;b) = 0$ – prirodni splajn,
(*ii*) $f(x)$ i $S_\Delta(f;x)$ su periodične funkcije na $[a,b]$ – periodični splajn
(*iii*)
$S'_\Delta(f;a) = f'(a), \quad S'_\Delta(f;b) = f'(b)$.

Svojstvo minimalnosti

$$\langle f\rangle^2 \geq \langle S_\Delta\rangle^2, \qquad \langle f\rangle^2 = \int_a^b \big(f''(x)\big)^2\,dx.$$

Trigonometrijski interpolacioni polinom

Za interpolaciju periodičnih funkcija
$f(x) = f(x+2k\pi), \quad k = 0,\pm1,\ldots$

$$f(x) \approx Q(x) = \sum_{i=0}^{n}\Bigg(\prod_{\substack{j=0\\ j\neq i}}^{n} \frac{\sin\,(x-x_j)}{\sin\,(x_i-x_j)}\Bigg) f(x_i)$$

2. Srednjekvadratna aproksimacija

$$f,\ g_0, \ldots, g_n \in \mathcal{L}_2[a,b], \quad \mathcal{L}_2[a,b] = \{f \mid \|f\| < \infty\}$$

$$(f,g) = \int_a^b p(x) f(x) g(x)\, dx, \quad \|f\|^2 = (f,f)$$

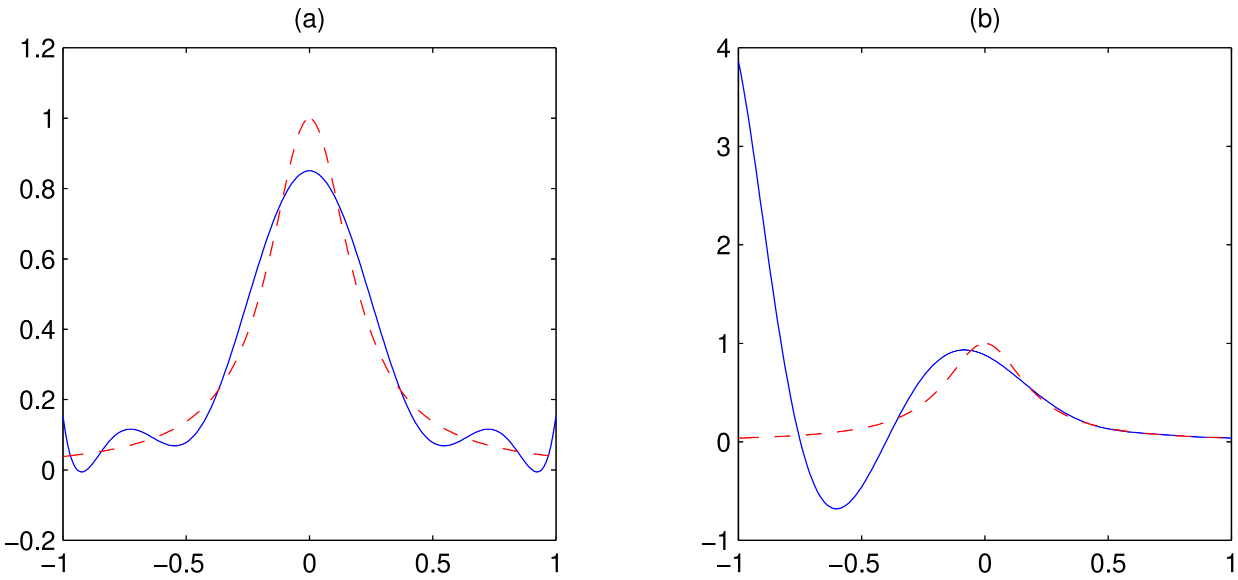
$p(x) > 0$ je težinska funkcija.

Srednjekvadratna aproksimacija
$Q_\circ(x) = \sum_{k=0}^n c_k^\circ g_k(x)$ funkcije $f(x)$
postoji i jedinstveno je određena,

$$E_n(f) = \left\| f - \sum_{k=0}^n c_k^\circ g_k \right\| = \inf_{c_0,\ldots,c_n} \left\| f - \sum_{k=0}^n c_k g_k \right\|$$

$E_n(f)$ je greška aproksimacije.

$$\|f - Q_\circ\| = \inf_Q \|f - Q\| = \inf_Q \left(\int_a^b p(f-Q)^2\, dx \right)^{1/2}$$



$p(x) \equiv 1$ $\qquad$ $p(x) = e^{10x}$

$Q_\circ$ je ortogonalna projekcija f na lineal $L(g_0, \ldots, g_n)$, jer je

$$(f - Q_\circ\,,\, Q) = 0 \quad \forall Q \in L(g_0, \ldots, g_n)$$

$$\sum_{k=0}^{n} c_k^\circ (g_k, g_m) = (f, g_m) \qquad m = 0, \ldots, n.$$

Ortonormirani sistem funkcija

$$(g_k, g_m) = \int_a^b p(x) g_k(x) g_m(x)\, dx = \delta(k - m),$$
$$k, m = 0, \ldots, n.$$

Ortogonalni polinomi

Legendre-ovi polinomi

$p(x) \equiv 1, \quad [a,b] \equiv [-1,1]$

$$L_m(x) = \frac{1}{2^m m!} \frac{d^m}{dx^m}\left((x^2-1)^m\right).$$

$$(L_k\,,\,L_m) = \begin{cases} 0, & k \neq m \\ \frac{2}{2m+1}, & k = m \end{cases}$$

Polinomi Čebiševa

$p(x) = \frac{1}{\sqrt{1-x^2}}, \quad [a,b] \equiv [-1,1]$

$$\begin{aligned} T_m(x) &= \cos(m \arccos x) \\ &= \frac{1}{2}\left(\left(x+\sqrt{x^2-1}\right)^m + \left(x-\sqrt{x^2-1}\right)^m\right) \end{aligned}$$

$$(T_k\,,\,T_m) \equiv \int_{-1}^{1} \frac{T_k(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \begin{cases} 0, & \text{za } m \neq k \\ \frac{\pi}{2}, & \text{za } m = k \neq 0 \\ \pi, & \text{za } m = k = 0. \end{cases}$$

Fourier-ov koeficijent $c_k^\circ = (f, g_k)$

$$f(x) \approx Q_\circ(x) = \textstyle\sum_{k=0}^{n} (f, g_k)\, g_k(x)$$

$$\big(E_n(f)\big)^2 = \|f - Q_\circ\|^2 = \|f\|^2 - \sum_{k=0}^{n} |(f, g_k)|^2$$

Bessel-ova nejednakost $\|f\|^2 \geq \sum_{k=0}^{n} |(f\ ,\ g_k)|^2$

Kada aproksimaciju, umesto na konačnom, određujemo na prebrojivom bazisu važi *Parseval-ova jednakost*

$$\|f\|^2 = \sum_{k=0}^{\infty} |c_k^\circ|^2, \quad c_k^\circ = (f, g_k)$$

Fourier-ov red (Joseph Fourier, 1807.g)

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

$$a_k = \frac{\big(f\,,\ \cos kt\big)}{\big(\cos kt\,,\ \cos kt\big)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt$$

$$b_k = \frac{\big(f\,,\ \sin kt\big)}{\big(\sin kt\,,\ \sin kt\big)} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt$$

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\imath kt},$$

$$c_k = \frac{\big(f\,,\ e^{\imath kt}\big)}{\big(e^{\imath kt}\,,\ e^{\imath kt}\big)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-\imath kt}\,dt$$

Parsevalova jednakost $\quad \|f\|^2 = \sum_{k=-\infty}^{\infty} |c_k|^2$

$\{e^{\imath kx}\}$ potpun ortogonalni sistem
funkcija, koje su sopstvene funkcije

operatora diferenciranja i operatora konačnih razlika

$$\frac{d}{dx}\, e^{\imath kx} = \imath k\, e^{\imath kx} \qquad \triangle\, e^{\imath kx} = \left(\frac{e^{\imath kh}-1}{h}\right) e^{\imath kx}$$

Fourier-ova transformacija

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-\imath\omega t}\, dt \qquad \text{direktna}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(\omega)e^{\imath\omega t}\, d\omega \qquad \text{inverzna}$$

Parsevalova jednakost u ovom slučaju je

$$2\pi \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |\widehat{f}(\omega)|^2\, d\omega.$$

Diskretna Fourier-ova transformacija

$$f\Big(\frac{2\pi j}{n}\Big) = \sum_{k=0}^{n-1} c_k e^{\imath k \frac{2\pi j}{n}}, \quad j = 0, \ldots, n-1.$$

Oznake: $W = e^{\imath \frac{2\pi}{n}} = \sqrt[n]{e^{\imath 2\pi}}, \quad f_j = f\big(\frac{2\pi j}{n}\big)$

$$\sum_{k=0}^{n-1} c_k W^{kj} = f_j, \quad j = 0, \ldots, n-1,$$

Fourier-ova matrica

$$F = \begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ 1 & W & W^2 & \ldots & W^{n-1} \\ 1 & W^2 & W^4 & \ldots & W^{2(n-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W^{n-1} & W^{2(n-1)} & \ldots & W^{(n-1)^2} \end{pmatrix},$$

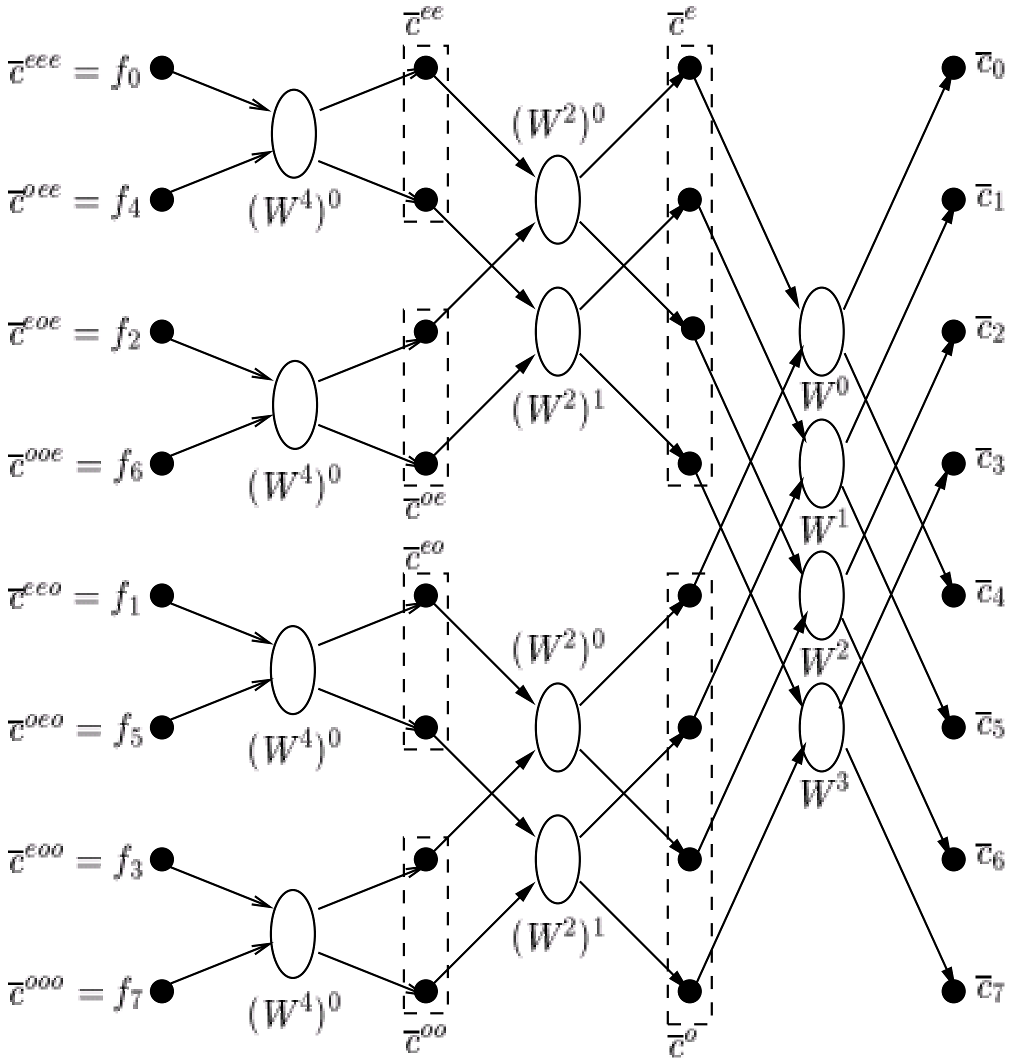
$$F\,F^* = F^*\,F = nI, \quad F\,\mathbf{c} = \mathbf{f}, \quad \frac{1}{n} F^*\,\mathbf{f} = \mathbf{c}$$

Brza Fourier-ova transformacija (FFT)

$$\mathbf{y} = F_n \mathbf{x}, \qquad \mathbf{y}^e = F_m \mathbf{x}^e, \quad \mathbf{y}^o = F_m \mathbf{x}^o \qquad n = 2m,$$

$$\mathbf{x}^e = (x_0, x_2, \ldots, x_{n-2})^T, \quad \mathbf{x}^o = (x_1, x_3, \ldots, x_{n-1})^T$$

$$\begin{aligned} y_j &= y_j^e + W_n^j y_j^o \\ y_{m+j} &= y_j^e - W_n^j y_j^o \end{aligned} \qquad j = 0, \ldots, m-1.$$

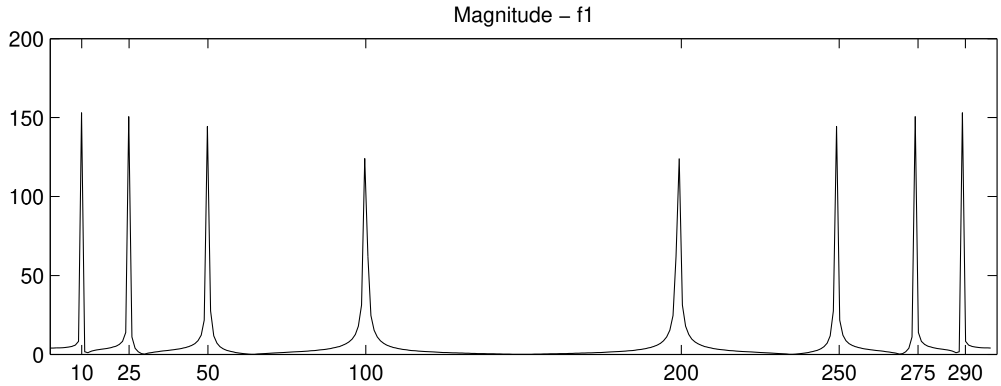


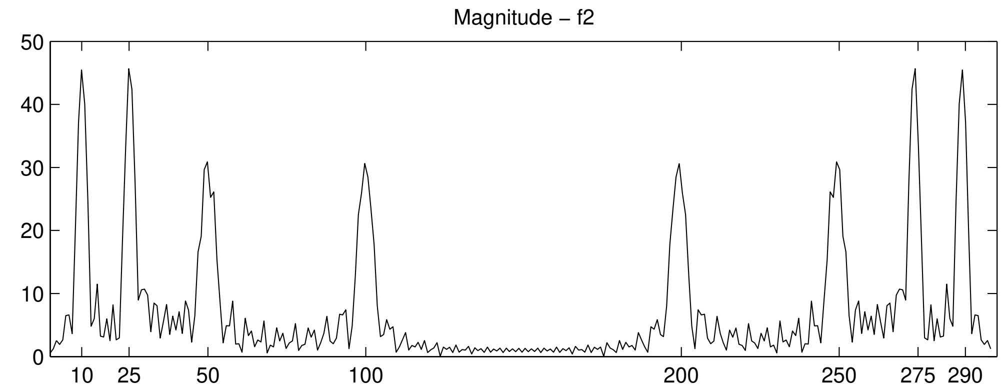
Butterfly struktura.

Nedostaci Fourierove analize

$$f1(t) = \cos(2\pi * 10 * t) + \cos(2\pi * 25 * t) + \cos(2\pi * 50 * t) + \cos(2\pi * 100 * t)$$

$$f2(t) = \begin{cases} \cos(2\pi * 10 * t), & 0 < t < 300 \\ \cos(2\pi * 25 * t), & 300 < t < 600 \\ \cos(2\pi * 50 * t), & 600 < t < 800 \\ \cos(2\pi * 100 * t), & 800 < t < 1000 \end{cases}$$





Prozor Fourierova transformacija
(Window Fourier Transformation,
STFT=ShortTimeFourierTransformation)

$$STFT_f(\omega,\tau) = \big(f(t)\,,\,g_{\omega,\tau}(t)\big)$$
$$= \int_{-\infty}^{\infty} f(t) w^*(t-\tau) e^{-\imath\omega t}\,dt$$

Prozor funkcija $\quad g_{\omega,\tau}(t) = w(t-\tau)e^{\imath\omega t}$

Na primer, za $w(t) = \begin{cases} 1, & t \in [0,1) \\ 0, & t \notin [0,1) \end{cases}$ je

$$STFT_f(\omega,\tau) = \widehat{f}(\omega), \quad t \in [\tau, \tau+1]$$

Nedostatak je fiksna širina prozora.

Transformacija talasićima
(WT=WaveletTransformation)

Direktna

$$CWT_f(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\,\psi^*\Big(\frac{t-b}{a}\Big)\,dt$$
$$= \big(f(t)\,,\,\psi_{a,b}(t)\big)$$

Inverzna

$$f(t) = \frac{1}{C_\psi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} CWT_f(a,b)\psi_{a,b}(t)\,\frac{da\,db}{a^2}$$

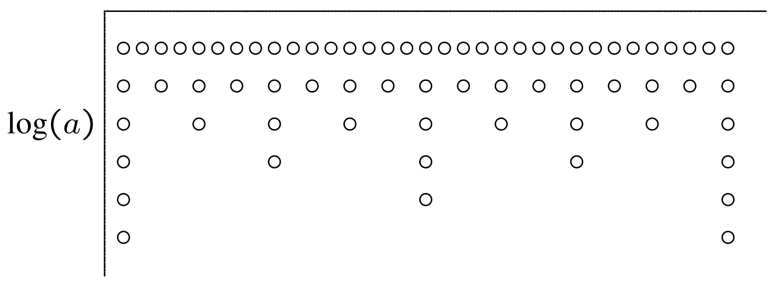
Talasić $\quad \psi_{a,b}(t) = \frac{1}{\sqrt{a}}\,\psi\left(\frac{t-b}{a}\right)$

Diskretni talasići

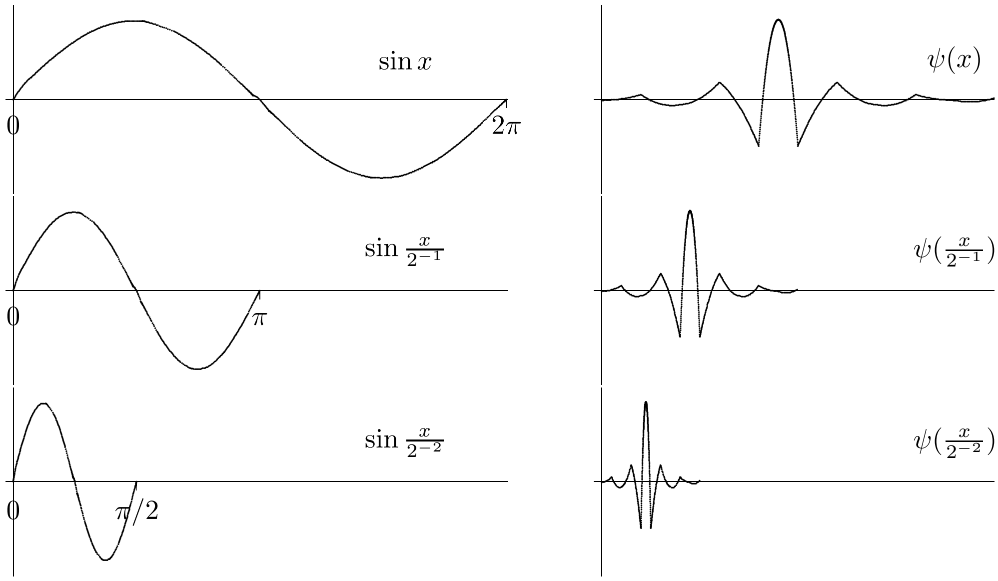
Diadsko samplovanje $a = 2^j,\ b = k\,2^j$

$$\psi_{jk}(t) = 2^{-j/2}\psi(2^{-j}t - k)$$

$$\psi_{jk}(t) \neq 0, \quad t \in [2^j k, 2^j(k+1)].$$



Multirezolucija

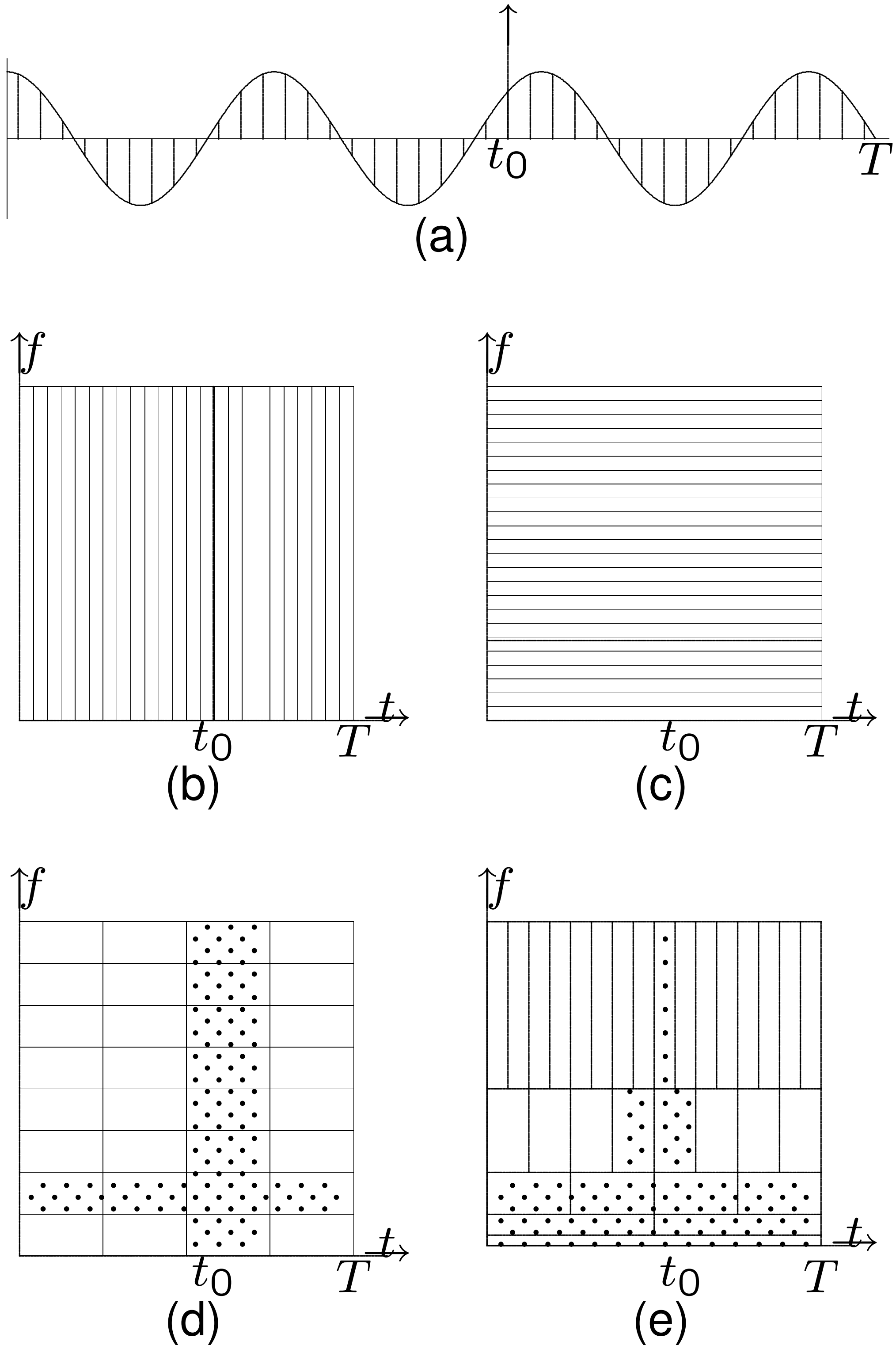


Fourierov bazis i bazis talasića

Primer:

$$s(t) = \sin(t) + \delta(t - t_0)$$

$$\delta(t) \sim \frac{1}{2\pi}(1 + 2\cos t + 2\cos 2t + \cdots)$$

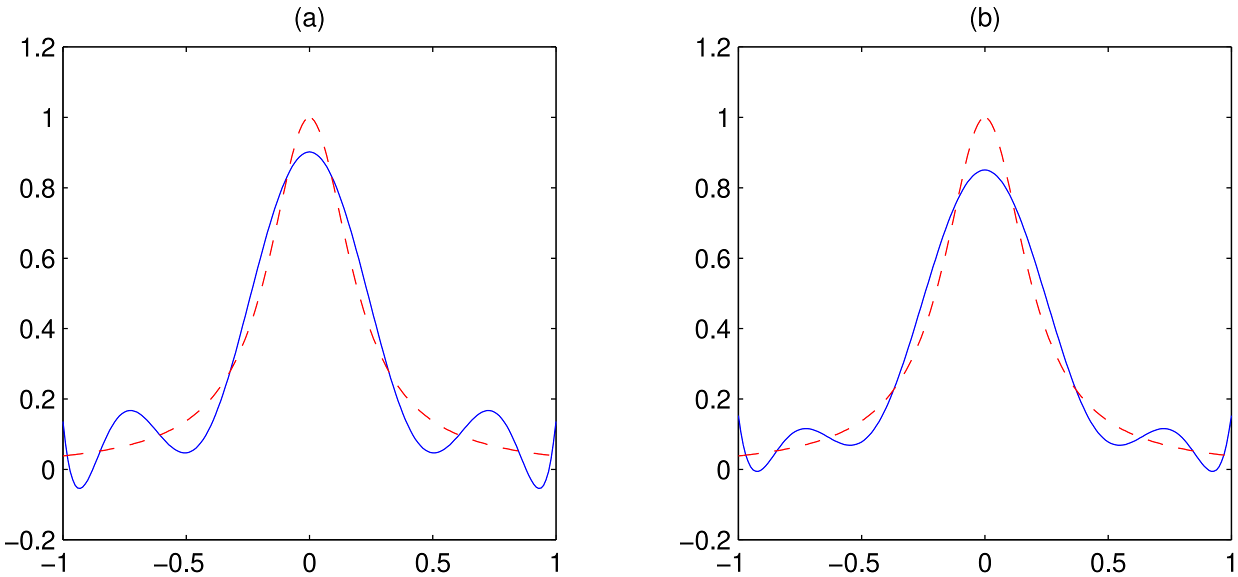


t_0
T
(a)
f
t
t_0
T
(b)
f
t
t_0
T
(c)
f
t
t_0
T
(d)
f
t
t_0
T
(e)

3. Ravnomerna aproksimacija

Na celom odsečku $[a, b]$ odstupanje funkcije $Q(x)$ od funkcije $f(x)$ ne sme da bude veće od dozvoljenog.

$$f(x) \in \mathcal{C}[a, b], \qquad \|f\| = \sup_{x \in [a,b]} |f(x)|$$



Ravnomerna (a) i srednjekvadratna (b) aproksimacija

Teorema Weierstrass-a: Ako je $f(x) \in \mathcal{C}[a,b]$, tada za proizvoljno $\varepsilon > 0$ postoji polinom $Q(x)$ takav da je

$$|f(x) - Q(x)| < \epsilon \qquad \forall x \in [a,b].$$

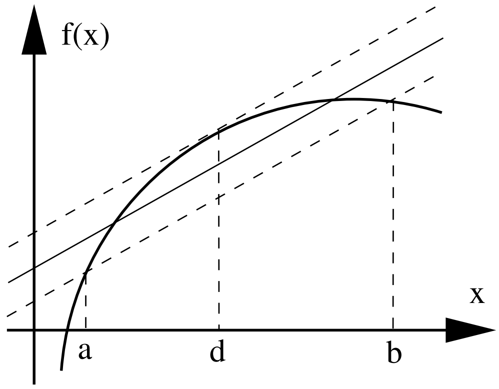
Teorema Čebiševa: Da bi polinom $Q(x)$ bio polinom najbolje ravnomerne aproksimacije neprekidne funkcije $f(x)$, potrebno je i dovoljno da na odsečku $[a,b]$ postoji bar $(n+2)$ tačaka $x_0 < \cdots < x_{n+1}$ takvih da je

$$f(x_i) - Q(x_i) = \alpha(-1)^i \|f - Q\|, \quad i = 0, \ldots, n+1,$$

pri čemu je $\alpha = 1$ ili $\alpha = -1$ istovremeno za sve i.

Tačke $x_0, \ldots, x_{n+1}$, koje zadovoljavaju uslove teoreme, nazivaju se *tačkama Čebiševljeve alternanse*.

Ako se konveksna funkcija ravnomerno aproksimira pravom, tačke Čebiševljeve alternanse su krajevi intervala i tačka ekstrema razlike funkcije i aproksimacije (a, b i d na slici)



Polinomi najbolje ravnomerne aproksimacije višeg stepena određuju se iterativnim algoritmom.

Polinomi Čebiševa imaju poseban značaj i u ravnomernoj aproksimaciji.

Ako je $P_n(x)$ proizvoljan polinom sa koeficijentom jedan uz najviši stepen, onda je

$$\max_{x\in[-1,1]} |P_n(x)| \geq \max_{x\in[-1,1]} |\overline{T}_n(x)| = 2^{1-n},$$
$$\overline{T}_n(x) = 2^{1-n} T_n(x)$$

Primer. Greška Lagrange-ove interpolacije će biti najmanja, ako se za čvorove interpolacije izaberu nule Čebiševljevog polinoma,

$$|f(x) - L_{n-1}(x)| = |\tfrac{1}{n!} f^{(n)}(\xi) \overline{T}_n^{[a,b]}(x)|$$
$$\leq \frac{1}{n!} \max_{x\in[a,b]} |f^{(n)}(x)| (b-a)^n 2^{1-2n}$$

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